Differential Equations Cheat Sheet

Let t be the *independent variable*, and x and y be *dependent variables*, where $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$.

For non-linear DEs, use *Separation of Variables*. For non-constant coefficients, use the *Integrating Factor*. Otherwise just use the *Ansatz Method*.

Tip: in acceleration problems, $a = v \times \frac{dv}{dx}$.

System of Differential Equations

1. Write $\dot{x}$ and $\dot{y}$ in terms of x and y.
2. Differentiate $\dot{x}$ and substitute $\dot{y}$.
3. Rearrange $\dot{x}$ for y and substitute y.
4. Use the *Ansatz Method* to find the general solution for x.
5. Substitute x and $\dot{x}$ to find the general solution for y.

Particular Integral

Polynomial	Polynomial of the same degree
Trigonometric	Cosine and sine of the same frequency
Exponential	Exponential of the same power

If the *particular integral* already appears in the *complementary function*, add a factor of t.

Ansatz Method

1. Write your equation in the form
$$a\ddot{x} + b\dot{x} + cx = f(t)$$
2. Solve the *auxiliary function*
$$a\lambda^2 + b\lambda + c = 0$$
3. Find the *complementary function*:
   - $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ for real roots.
   - $x = Ae^{\lambda t} + Bte^{\lambda t}$ for repeated roots.
   - $x = e^{\Re(\lambda)}(A\cos(\Im(\lambda)t) + B\sin(\Im(\lambda)t))$ for complex roots.
4. Guess and check the *particular integral*.
5. The *general solution* is $x = \text{CF} + \text{PI}$.
6. To find the *particular solution*, solve for A and B using initial conditions of t, x, and $\dot{x}$.

Recurrence Relations

1. Write your recurrence relation in the form
$$au_{n+2} + bu_{n+1} + cu_n = f(n)$$
2. Solve the *auxiliary function*
$$a\lambda^2 + b\lambda + c = 0$$
3. Find the *complementary function*:
   - $u_n = A\lambda_1^n + B\lambda_2^n$ for real roots.
   - $u_n = A\lambda^n + Bn\lambda^n$ for repeated roots.
   - $u_n = |\lambda|^n \ (A\cos(\arg(\lambda)n) + B\sin(\arg(\lambda)n))$ for complex roots.
4. Guess and check the *particular function*.
5. The *general solution* is $x = \text{CF} + \text{PF}$.
6. To find the *particular solution*, solve for A and B using initial values of u.

Integrating Factor

1. Write your equation in the form
$$\dot{x} + p(t)x = q(t)$$
2. Multiply both sides by the unknown *integrating factor* $I(t)$.
3. Rewrite the left hand side using the product rule to the derivative of $I(t)x$.
4. Find $I(t) = e^{\int p(t)dt}$.
5. Integrate both sides with respect to t.
6. Make x the subject.

Separation of Variables

1. Write your equation in the form
$$h(x)\dot{x} = g(t)$$
2. Integrate both sides with respect to t.
3. Make x the subject.

Complex Numbers Cheat Sheet

Complex numbers can be written in two forms

$$r(\cos\theta + i\sin\theta) = re^{\theta i}$$

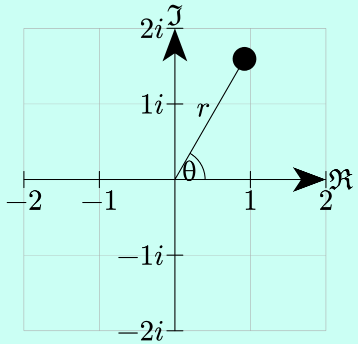
where $r, \theta \in \mathbb{R}$.

De Moivre's Theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where $\theta \in \mathbb{R}, n \in \mathbb{Z}$.

Complex numbers can be represented on an Argand diagram.



Complex Roots

A complex number z has n complex n^{th} roots ω.

$$\omega^n = z \quad \Rightarrow \quad \omega = \sqrt[n]{|z|}e^{\frac{\arg(z)+2\pi k}{n}i}$$
$$k \in \mathbb{N}, 0 \le k < n$$

The complex roots form a regular n-sided polygon on an Argand diagram.

De Moivre: Power to Frequency

1. Substitute $\cos\theta = \frac{e^{\theta i}+e^{-\theta i}}{2}$ or $\sin\theta = \frac{e^{\theta i}-e^{-\theta i}}{2i}$.
2. Expand binomially and collect like terms.
3. Substitute $e^{n\theta i} + e^{-n\theta i} = 2\cos\theta$ or $e^{n\theta i} - e^{-n\theta i} = 2i\sin\theta$.

De Moivre: Binomial C+iS

1. If you are given S, write C in a similar form. If you are given C, write S in a similar form.
2. Write $C + iS$ with each term in Euler form.
3. Rewrite using the Binomial expansion

$$(1+z)^n = 1 + \binom{n}{1}z + \binom{n}{2}z^2 + \ldots + z^n$$

4. Factorise $z^{\frac{n}{2}}$ and rewrite it in cartesian form.
5. Substitute $e^{n\theta i} + e^{-n\theta i} = 2\cos\theta$ or $e^{n\theta i} - e^{-n\theta i} = 2i\sin\theta$.
6. Simplify $C = \Re(C + iS)$ and $S = \Im(C + iS)$.

Complex Loci

There are three main types of complex loci.

1. Circles: $|z - (a + bi)| = r$
2. Lines: $|z - (a + bi)| = |z - (c + di)|$
3. Rays: $\arg(z - (a + bi)) = \theta$

De Moivre: Frequency to Power

1. Substitute $\cos n\theta = \Re((\cos\theta + i\sin\theta)^n)$ or $\sin n\theta = \Im((\cos\theta + i\sin\theta)^n)$.
2. Expand binomially and take the real or imaginary terms.
3. Apply $\sin^2\theta + \cos^2\theta = 1$ if needed.

De Moivre: Geometric C+iS

1. If you are given S, write C in a similar form. If you are given C, write S in a similar form.
2. Write $C + iS$ with each term in Euler form.
3. Rewrite using the geometric sum formula

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

4. Realise the denominator. *Remember*: change the sign in the exponent, not the coefficient.
5. Substitute $e^{n\theta i} + e^{-n\theta i} = 2\cos\theta$ or $e^{n\theta i} - e^{-n\theta i} = 2i\sin\theta$.
6. Simplify $C = \Re(C + iS)$ and $S = \Im(C + iS)$.

Complex Conjugates

The complex conjugate of a complex number negates the imaginary part, thus reflecting it in the real axis of an argand diagram.

$$z = a + bi \quad \Rightarrow \quad z^* = a - bi$$

Vectors Cheat Sheet

A vector represents a point or a displacement.

Lines

A line can be described in terms of a position vector and a direction vector.

$$\vec{r} = \vec{a} + \lambda \vec{d}$$

It can also be written using Cartesian coordinates.

$$\frac{x - a_x}{d_x} = \frac{y - a_y}{d_y} = \frac{z - a_z}{d_z}$$

Planes

A plane can be described in terms of a position vector and two direction vectors.

$$\vec{r} = \vec{a} + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2$$

A plane can be described in terms of a position vector and a normal vector.

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

It can also be written using Cartesian coordinates.

$$ax + by + cz + d = 0$$

Vector Products

The *dot product* produces a scalar.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

The *cross product* produces a vector perpendicular to both vectors.

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$$

$$\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \times \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

Three Planes Problems

Two planes are parallel if and only if their normal vectors are scalar multiples of each other.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -d_1 \\ -d_2 \\ -d_3 \end{pmatrix}$$

Three non-parallel planes can intersect...

1. At a point - if the matrix is invertible.
2. Forming a sheaf (book) - if each pair of planes intersects along a different line.
3. Forming a triangular prism - if all three planes intersect along the same line.

Calculating Distances

	Point	**Line**	**Plane**
Point	$\lvert\vec{p}_1 - \vec{p}_2\rvert$	$\boxed{\frac{\lvert(\vec{p} - \vec{a}) \times \vec{d}\rvert}{\lvert\vec{d}\rvert}}$	$\frac{\lvert(\vec{p} - \vec{a}) \cdot \vec{n}\rvert}{\lvert\vec{n}\rvert}$
Line	–	$\left\lvert \frac{\vec{d}_1 \times \vec{d}_2}{\lvert\vec{d}_1 \times \vec{d}_2\rvert} \cdot (\vec{a}_1 - \vec{a}_2) \right\rvert$	Any point to plane
Plane	–	–	Any point to plane

Note: point to line is not given on the formula sheet!

Matrices Cheat Sheet

A matrix represents a transformation.

Matrices can be multiplied together to compose transformations. Note that the order matters.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

The identity matrix I represents the transformation of not doing anything.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus for any matrix M

$$MI = IM = M$$

Determinant of a Matrix

For 2x2 matrices, the determinant is the area of the unit square after it has been transformed into a parallelogram.

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

For 3x3 matrices, the determinant is the volume of the unit cube after it has been transformed into a parallelepiped.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix} \cdot \begin{pmatrix} b \\ e \\ h \end{pmatrix} \times \begin{pmatrix} c \\ f \\ i \end{pmatrix}$$

Inverse of a Matrix

A matrix M has inverse M^{-1} if and only if

$$MM^{-1} = M^{-1}M = I$$

A matrix only has an inverse if it is a square matrix with a non-zero determinant.

For 2x2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For 3x3 matrices

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}}\begin{bmatrix} (\vec{v_2} \times \vec{v_3})^T \\ (\vec{v_3} \times \vec{v_1})^T \\ (\vec{v_1} \times \vec{v_2})^T \end{bmatrix}$$

Eigenvalues

For a matrix M, its *eigenvalues* λ are the solutions to its *characteristic equation*

$$\det(M - \lambda I) = 0$$

Each *eigenvalue* represents the scale factor of an *invariant line* when the matrix is applied.

Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem states that if you substitute a matrix into its *characteristic equation*, the result will be zero.

Eigenvectors

For an *eigenvalue* λ, its *eigenvectors* $\vec{v}$ are given by the equation

$$(M - \lambda I)\vec{v} = 0$$

The set of all eigenvectors form an *invariant line*, also known as an *eigenspace*.

As all eigenvectors are scalar multiples of each other, it is often enough to let $\vec{v} = \begin{pmatrix} 1 \\ a \end{pmatrix}$ and solve.

For 3x3 matrices, you quickly find an *eigenvector* by taking the cross product of two rows in the matrix $M - \lambda I$ (as long as they are not parallel).

Diagonalisation

If you know the eigenvalues λ_1, λ_2 and eigenvectors $\vec{v_1}, \vec{v_2}$ of a matrix M, you can use a change of base to *diagonalise* the matrix into a simple enlargement along each axis.

$$M = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix}^{-1}$$

This also works for 3x3 matrices.

$$M = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}^{-1}$$

Multivariable Calculus Cheat Sheet

A partial derivative is a lazy derivative. When taking the partial derivative with respect to x, everything else is treated as a constant.

$$\frac{\partial}{\partial x}(x^2 + xy + y^2) = 2x + y$$

Curved Surfaces

There are two ways to define surfaces.

1. Explicit: $z = f(x, y)$ maps each pair of x, y values to a unique z value. This only works if the surface has no holes or arches.
2. Implicit: $g(x, y, z) = 0$ represents the roots of a four-dimensional surface. This allows us to describe a wider range of surfaces.

Contours and Sections

A *contour* is the cross-section we get when we choose a constant value of z.

A *section* is the cross-section we get when we choose a constant value of x or y.

Turning Points

A point on an explicit surface is a turning point if and only if it is a turning point in the x-section and the y-section.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

Finding the nature of the turning point is non-trivial. Exam questions are typically looking for simple algebraic or geometric reasoning.

Grad

The vector ∇g represents the *direction and rate of fastest increase* of the surface at a given point. It is always normal to the surface.

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix}$$

If $g(x, y, z) = f(x, y) - z$ there is a special case.

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ -1 \end{pmatrix}$$

Tangent Plane and Normal Line

The tangent plane to a surface g at a point $\vec{a}$ is given by the usual plane equation

$$(\vec{r} - \vec{a}) \cdot \nabla g = 0$$

where ∇g is evaluated at $\vec{a}$.

Likewise, the normal line to a surface g at a point $\vec{a}$ is given by the usual line equation

$$\vec{r} = \vec{a} + \lambda(\nabla g)$$

where ∇g is evaluated at $\vec{a}$.

Groups Cheat Sheet

A group is defined as a set S and a binary operation $\star$ which acts on the elements of S.

A group must satisfy the following four axioms:

1. *Closure*: $\forall a, b \in S$
$$a \star b \in S$$
2. *Identity*: $\exists e \in S, \forall a \in S$
$$a \star e = e \star a = a$$
3. *Invertibility*: $\forall a \in S, \exists a^{-1} \in S$
$$a \star a^{-1} = a^{-1} \star a = e$$
4. *Associability*: $\forall a, b, c \in S$
$$a \star (b \star c) = (a \star b) \star c$$

Cayley Tables

We can represent a group in a Cayley table. The table below shows the K_4 group.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The order of operations is top $\star$ left. The *latin square property* means that each element will appear exactly once in each row and column of the Cayley table. This is true for all groups but not an axiom.

Small Groups

Groups of order 3:
- $C_3 = (\{0, 1, 2\}, +_3)$

Groups of order 4:
- $C_4 = (\{0, 1, 2, 3\}, +_4)$
- $K_4 = (\text{symmetries of } ▭, \circ)$

Groups of order 5:
- $C_5 = (\{0, 1, 2, 3, 4\}, +_5)$

Groups of order 6:
- $C_6 = (\{0, 1, 2, 3, 4, 5\}, +_6)$
- D_6 or $S_3 = (\text{symmetries of } \triangle, \circ)$

Subgroups

For a given group $(S, \star)$, another group $(T, \star)$ is a subgroup if $T \subset S$.
- The subgroup is *trivial* if T only has one element (the identity).
- The subgroup is *improper* if $T = S$ (literally the same group).

Order

The *order of a group* is the number of elements in the set that makes up the group.

The *order of an element* a is the smallest positive integer n such that $a^n = e$.

Lagrange's Theorem

The order of a subgroup must divide the order of the group.
- The order of an element must divide the order of the group.
- Any group of prime order must be cyclic.

Integration Cheat Sheet

The most difficult part of integration is figuring out which method to use. Just remember PROP.

- **P**artial fractions: try to split your integral into two easier integrals.
- **R**everse chain rule: try to guess what might have been differentiated to give the integral, especially $\ln|f(x)|$.
- **O**pt for substitution: try to guess what to substitute to simplify the integral, especially using trigonometric identities.
- **P**arts: try to integrate by parts, and consider if one of the parts is just 1.

Integration by Substitution

Let $x = g(u) \Rightarrow \frac{dx}{du} = g'(u)$.

You can rewrite your integral as

$$\int f(x)dx = \int f(g(u))g'(u)du$$

Note that this is essentially a more rigorous method for reverse chain rule.

Integration by Parts

You can rewrite your integral as

$$\int u\dot{v}dx = uv - \int \dot{u}vdx$$

Integrals

$f(x)$	$\int f(x)dx - C$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\cos x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\frac{x}{a}$
$-\frac{1}{\sqrt{a^2 - x^2}}$	$\arccos\frac{x}{a}$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan\frac{x}{a}$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh}\frac{x}{a}$
$\frac{1}{\sqrt{-a^2 + x^2}}$	$\operatorname{arcosh}\frac{x}{a}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{a}\operatorname{artanh}\frac{x}{a}$

Miscellaneous

The average value of $f(x)$ from a to b is given by

$$\frac{1}{b-a}\int_a^b f(x)dx$$

The volume of revolution from a to b is given by

$$\int_a^b \pi y^2 dx$$

The area of a polar curve from a to b is given by

$$\int_a^b \frac{1}{2}r^2 d\theta$$

More Integrals

$f(x)$	$\int f(x)dx - C$
$g(x)^n$	$\frac{g(x)^{n+1}}{(n+1)g'(x)}$
$e^{g(x)}$	$\frac{e^{g(x)}}{g'(x)}$
$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$